

A CLOSED FORM FOR UNITONS

CHRISTOPHER KUMAR ANAND

McGill University and University of Warwick

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ABSTRACT. Unitons, i.e. harmonic spheres in a unitary group, correspond to ‘uniton bundles’, i.e. holomorphic bundles over the compactified tangent space to the complex line with certain triviality and other properties. In this paper, we use a monad representation similar to Donaldson’s representation of instanton bundles to obtain a simple formula for the unitons. Using the monads, we show that real triviality for uniton bundles is automatic. We interpret the uniton number as the ‘length’ of a jumping line of the bundle, and identify the uniton bundles which correspond to based maps into Grassmannians. We also show that energy-3 unitons are 1-unitons, and give some examples.

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1. INTRODUCTION

Harmonic maps between Riemannian manifolds M and N are critical values of an energy functional

$$\text{energy}(S : M \rightarrow N) = \frac{1}{2} \int_M |dS|^2.$$

In the case of surfaces in a matrix group, with the standard (left-invariant) metric, the energy takes the form

$$(1.1) \quad \text{energy}(S) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|S^{-1} \frac{\partial}{\partial x} S|^2 + |S^{-1} \frac{\partial}{\partial y} S|^2 \right) dx \wedge dy.$$

Unitons are harmonic maps $S : \mathbb{S}^2 \rightarrow \text{U}(N)$. Some authors call them multi-unitons. Since the energy is conformally invariant in the case of surfaces, it is natural to use coordinates x and y on \mathbb{R}^2 (or $z \in \mathbb{C}$) and derive the Euler-Lagrange or unton equations,

$$(1.2) \quad \frac{\partial}{\partial x} (S^{-1} \frac{\partial}{\partial x} S) + \frac{\partial}{\partial y} (S^{-1} \frac{\partial}{\partial y} S) = 0.$$

From [SaUhl, Theorem 3.6], we know that harmonic maps from $\mathbb{R}^2 \rightarrow \text{U}(N)$ extend to \mathbb{S}^2 iff they have finite energy, and that such maps are always smooth. So working in terms of coordinates x and y on \mathbb{R}^2 or $z \in \mathbb{C}$ poses no real limitation.

1.3 Based Unitons. Unitons are determined by the pullback of the Maurer-Cartan form on $\text{U}(N)$,

$$(1.4) \quad A \stackrel{\text{def}}{=} \frac{1}{2} S^{-1} dS = A_x dx + A_y dy = A_z dz + A_{\bar{z}} d\bar{z}$$

and a choice of a basepoint, $S(\infty) \in \text{U}(N)$, as we can see by thinking of $d + 2A$ as a flat connection and S as a gauge transformation. (See [An1].)

We are concerned with the based maps

$$\text{Harm}_k^*(\mathbb{S}^2, \text{U}(N)) \stackrel{\text{def}}{=} \{S \in \text{Harm}(\mathbb{S}^2, \text{U}(N)) : S(\infty) = 1, \text{energy}(S) = k\}.$$

In [Uhl], Uhlenbeck showed that all unitons could be constructed from simpler unitons by ‘adding a unton’. This construction was investigated, from different perspectives, by Wood, Valli, Guest, Ohnita and Segal. We approach the question of constructing unitons and investigating their moduli via algebraic integration, using a twistor construction of Hitchin and Ward ([Hi],[Wa]). We proved in [An1] that the based unitons, $\text{Harm}^*(\mathbb{S}^2, \text{U}(N))$, are isomorphic to unton bundles, with energy corresponding to the bundles’ second Chern class. Given transition matrices for the unton bundle, we showed how to construct solutions. This equivalence is explained in §2.

In this paper we apply Horrocks’ monad construction to the unton bundles, i.e. we show that unton bundles are representable as the cohomology of a short sequence (of homogeneous bundles on \mathbb{P}^2 in this case). After interpreting the reality and triviality properties in §3, we derive the main result of this paper,

Theorem A. *Based, rank- N unitons of energy $8\pi k$ are all of the form*

$$S = \mathbb{I} + a\alpha_2^{-1}(\alpha_1 - 2(x + iy\alpha_2))^{-1}b$$

for some choice of $N \times k$, $k \times N$, $k \times k$ and $k \times k$ matrices, a , b , α_1 , α_2 . (Multiplication is matrix multiplication.)

We do this by specialising the ‘monodromy’ interpretation of Uhlenbeck’s extended solution (see §2.11) to bundles given by monads. In §5 we explain how to obtain the monad representation for the example of a 2-uniton in [An1]. In §6, we show that the uniton number is also the length of the jumping line which determines the uniton bundle. We prove in the next section that real triviality is implied by the other bundle properties. This is a geometric fact whose proof depends on the monad representation. We complete the GIT picture in §8 by representing time invariance with its infinitesimal generator. As a corollary, we give a simple finite-gap result. The next two sections work out the 1-uniton case in general. Finally, in §11, we explain how to identify those solutions which factor through the Cartan embedding of a Grassmannian. Precisely, we have

Theorem B.

- (1) *The space of based unitons $\text{Harm}_k^*(\mathbb{S}^2, \text{U}(N))$ is isomorphic to the set of monad data*

$$\begin{aligned} \gamma, \alpha'_1, \delta &\in \mathfrak{gl}(k/2), \quad \gamma \text{ nilpotent} \\ a' &\in \text{M}_{N, k/2}, \quad b' \in \text{M}_{k/2, N} \end{aligned}$$

satisfying

(nondegeneracy)

$$\text{rank} \begin{pmatrix} \gamma \\ \alpha'_1 + z \\ a' \end{pmatrix} = \text{rank} \begin{pmatrix} \gamma & \alpha'_1 + z & b' \end{pmatrix} = k/2 \quad \forall z \in \mathbb{C}$$

(monad equation)

$$[\alpha'_1, 2\gamma] + b'a' = 0$$

(time invariance)

$$\begin{aligned} [\delta, \gamma] &= 0 & a'\delta &= 0 \\ [\delta, \alpha'_1] &= \gamma & \delta b' &= 0 \end{aligned}$$

quotiented by the action of $g \in \text{Gl}(k/2)$

(group action)

$$\begin{aligned} \gamma &\mapsto g\gamma g^{-1} & \alpha'_1 &\mapsto g\alpha'_1 g^{-1} & \delta &\mapsto g\delta g^{-1} \\ a' &\mapsto a'g^{-1} & b' &\mapsto gb'. \end{aligned}$$

- (2) *These data determine the uniton bundle over a hemisphere. Reality determines it over the other hemisphere, giving monad data as in Theorem A as follows:*

$$\begin{aligned} \alpha_1 &= 2 \begin{pmatrix} \alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix} & \alpha_2 &= \begin{pmatrix} -\mathbb{I} - 2\gamma^* & \\ & \mathbb{I} + 2\gamma \end{pmatrix} \\ a &= 2 \begin{pmatrix} b'^* & a' \end{pmatrix} & b &= 2 \begin{pmatrix} a'^* \\ b' \end{pmatrix} \end{aligned}$$

where ϕ_1 and ϕ_2 are functions of γ , a' and b' determined by the big monad equation $[\alpha_1, \alpha_2] + ba = 0$.

- (3) The unton number (see §6) is the smallest $n \in \mathbb{Z}$ such that $\gamma^n = 0$.
- (4) The unton is in the image of $\text{Harm}_k^*(\mathbb{S}^2, \text{Gr}(\mathbb{C}^N)) \hookrightarrow \text{Harm}_k^*(\mathbb{S}^2, \text{U}(N))$ iff there is a $G \in \text{Gl}(k/2)$, and an $F \in \text{U}(N)$ such that $F^2 = \mathbb{I}$ and

$$\begin{aligned} G\alpha_2^{-1}\alpha_1G^{-1} &= \alpha_1 & Fa\alpha_2^{-1}G^{-1} &= a \\ G\alpha_2G^{-1} &= \alpha_2 & -G\alpha_2^{-2}bF^{-1} &= b. \end{aligned}$$

Clearly Theorem B can be used to investigate the topology of the moduli space of unitons (see [An2] for some results in low dimensions), but the details of such calculations properly belong in a separate paper. In this paper, we restrict ourselves to showing that the space of unton bundles has high codimension as a subspace of the space of framed jumping lines: generic nilpotent γ are not allowed. This implies that 2-unitons must have normalised energy 4 or more. It would be interesting to formulate and prove a more general finite-gap result of this type.

2. PREREQUISITES

2.1 Unton Bundles. The unton bundles are bundles on $\widetilde{T\mathbb{P}^1} \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$, the fibrewise compactification of the tangent bundle $T\mathbb{P}^1$ of the *complex* projective line.

Let (λ, η) and $(\hat{\lambda} = 1/\lambda, \hat{\eta} = \eta/\lambda^2)$ be coordinates on $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$, where λ is the usual coordinate on \mathbb{P}^1 and η is the coordinate associated to $d/d\lambda$. Meromorphic sections (s) of $T\mathbb{P}^1$ give all the holomorphic sections of $\widetilde{T\mathbb{P}^1}$ ($[s, 1]$ in projective coordinates on $\widetilde{T\mathbb{P}^1}$), save one. We fix notation for the lines on $\widetilde{T\mathbb{P}^1}$:

$$\begin{aligned} P_\lambda &= \pi^{-1}(\lambda \in \mathbb{P}^1) = \text{a pfibre (silent p)} \\ G_0 &= \{(\lambda, [0, 1])\} = \text{(graph of) zero section of } T\mathbb{P}^1 \\ G_\infty &= \{(\lambda, [1, 0])\} = \text{infinity section of } \widetilde{T\mathbb{P}^1} \\ G_{\eta=s} &= \{(\lambda, [s(\lambda), 1])\}. \end{aligned} \tag{2.2}$$

If $y = (a, b, c) \in \mathbb{C}^3$, we will also write G_y for $G_{\eta=\frac{1}{2}(a-2b\lambda-c\lambda^2)}$.

To encode unitarity, we need the real structure

$$\sigma^*(\lambda, \eta) = (1/\bar{\lambda}, -\bar{\lambda}^{-2}\bar{\eta}) \tag{2.3}$$

which acts by

$$\sigma^*(a, b, c) = (\bar{c}, -\bar{b}, \bar{a})$$

on $\mathbb{C}^3 \cong H^0(\mathbb{P}^1, \mathcal{O}(2))$, the space of finite sections. We similarly define time translation

$$\begin{aligned} \delta_t : (\lambda, \eta) &\mapsto (\lambda, \eta - 2t\lambda) \\ (a, b, c) &\mapsto (a, b + t, c). \end{aligned} \tag{2.4}$$

Definition 2.5. A rank N , or $U(N)$, uniton bundle, \mathcal{V} , is a holomorphic rank N bundle on $\widetilde{T\mathbb{P}^1}$ which is a) trivial when restricted to the following curves in $\widetilde{T\mathbb{P}^1}$

- (1) the section at infinity
- (2) nonpolar fibres (i.e. fibres above $\lambda \in \mathbb{C}^* \subset \mathbb{P}^1$)
- (3) real sections of $T\mathbb{P}^1$ (sections invariant under σ)

b) is equipped with bundle lifts

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\tilde{\delta}_t} & \mathcal{V} \\ \downarrow & & \downarrow \\ \widetilde{T\mathbb{P}^1} & \xrightarrow{\delta_t} & \widetilde{T\mathbb{P}^1} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{\tilde{\sigma}} & \mathcal{V}^* \\ \downarrow & & \downarrow \\ \widetilde{T\mathbb{P}^1} & \xrightarrow{\sigma} & \widetilde{T\mathbb{P}^1} \end{array}$$

- (1) $\tilde{\delta}_t$ a one-parameter family of holomorphic transformations fixing \mathcal{V} above the section at infinity, lifting δ_t , and
- (2) $\tilde{\sigma}$ a norm-preserving, antiholomorphic lift of σ such that the induced hermitian metric on \mathcal{V} restricted to a fixed point of σ is positive definite; equivalently, such that the induced lift to the principal bundle of frames acts on fibres of fixed points of σ by $X \mapsto X^{*-1}$.

and c) has a framing, $\phi \in H^0(P_{-1}, \text{frames}(\mathcal{V}))$, of the bundle \mathcal{V} restricted to the fibre $P_{-1} = \{\lambda = -1\} \subset \widetilde{T\mathbb{P}^1}$ such that $\tilde{\sigma}(\phi) = \phi$.

2.6 Extended Solutions. Uhlenbeck's extended solutions E_λ (actually first employed in [Po]), encode the unitons as follows

Theorem 2.7 [Uhl, 2.1]. Let $\Omega \subset \mathbb{S}^2$ be a simply-connected neighbourhood and $A : \Omega \rightarrow T^*(\Omega) \otimes \mathfrak{u}(N)$. Then $2A = S^{-1}dS$, with S harmonic iff the curvature of the connection

$$(2.8) \quad \mathcal{D}_\lambda = \left(\frac{\partial}{\partial \bar{z}} + (1 + \lambda)A_{\bar{z}}, \frac{\partial}{\partial z} + (1 + \lambda^{-1})A_z \right)$$

vanishes for all $\lambda \in \mathbb{C}^*$.

Theorem 2.9 [Uhl, 2.2]. If S is harmonic and $S(\infty) = \mathbb{I}$, then there exists a unique holomorphic family of covariant constant frames $E_\lambda : \mathbb{S}^2 \rightarrow U(N)$ for the connection \mathcal{D}_λ for each $\lambda \in \mathbb{C}^*$ with

- (1) $E_{-1} = \mathbb{I}$,
- (2) $E_1 = S$,
- (3) $E_\lambda(\infty) = \mathbb{I}$.

Moreover, E_λ is analytic and holomorphic in $\lambda \in \mathbb{C}^*$.

Theorem 2.10 [Uhl, 2.3]. Suppose $E : \mathbb{C}^* \times \Omega \rightarrow \text{Gl}(N)$ is analytic and holomorphic in the first variable, $E_{-1} \equiv \mathbb{I}$, and the expressions

$$\frac{E_\lambda^{-1} \bar{\partial} E_\lambda}{1 + \lambda}, \quad \frac{E_\lambda^{-1} \partial E_\lambda}{1 + \lambda^{-1}}$$

are constant in λ , then $S = E_1$ is harmonic.

The key to understanding the ‘monodromy’ interpretation of E_λ which we require from [An1] is to think of E_λ as a singular change of frame between the usual (constant) frame of \mathbb{C}^N and the trivialising frame for \mathcal{D}_λ which agrees with first frame at $z = \infty$.

2.11 Monodromy construction. The uniton bundle $\mathcal{V} \rightarrow \widetilde{T\mathbb{P}^1}$ is constructed generically as the kernel of a differential operator on $\mathbb{R}^3 \times \mathbb{S}^2$ ([An1]). The ‘connection’ \mathcal{D}_λ is the projection of this operator from $\mathbb{R}^3 \times \mathbb{S}^2$ to $\mathbb{R}^2 \times \mathbb{C}^*$. Pulling E_λ back to $\mathbb{R}^3 \times \mathbb{C}^*$ gives a solution to the defining operator on $(\mathbb{S}^2 \times \mathbb{R}) \times \mathbb{C}^*$ which gives a trivialisation of \mathcal{V} restricted to the open set $\{\lambda \in \mathbb{C}^*\}$. Intrinsically, it is the trivialisation on each fibre P_λ , $\lambda \in \mathbb{C}^*$, which agrees with a trivialisation of $\mathcal{V}|_{G_\infty}$. We require this trivialisation to agree with the framing ϕ at P_{-1} . By construction, the usual frame above a point $y \in \mathbb{R}^3$ also lifts to give a solution of the operator, this time giving a frame over the section G_y . This frame is also uniquely defined by compatibility with the framing. The change of frame E_λ can be computed by composing the cycle of isomorphisms

$$\begin{array}{ccccc}
 \mathcal{V}_{\lambda, \infty} & \xleftarrow{\text{eval}} & H^0(G_\infty, \mathcal{V}) & \xrightarrow{\text{eval}} & \mathcal{V}_{-1, \infty} \\
 \text{eval} \uparrow & & & & \uparrow \text{eval} \\
 (2.12) \quad H^0(P_\lambda, \mathcal{V}) & & & & H^0(P_{-1}, \mathcal{V}) \\
 \text{eval} \downarrow & & & & \downarrow \text{eval} \\
 \mathcal{V}_{(\lambda, z/2 - t\lambda - \lambda^2 \bar{z}/2)} & \xleftarrow{\text{eval}} & H^0(G_{(z, \bar{z}, t)}, \mathcal{V}) & \xrightarrow{\text{eval}} & \mathcal{V}_{(-1, z/2 - t\lambda - \bar{z}/2)}
 \end{array}$$

counter clockwise. The existence of the bundle isomorphism $\tilde{\delta}_t$ (time translation) ensures that the result doesn’t depend on t . Finiteness, i.e. extension to \mathbb{S}^2 , follows from the compactness of $\widetilde{T\mathbb{P}^1}$.

3. UNITON BUNDLES AND \mathbb{P}^2 MONADS

Uniton bundles are bundles over $\widetilde{T\mathbb{P}^1}$, the \mathbb{P}^1 bundle over \mathbb{P}^1 with a double twist. In [An1], the geometry of $\widetilde{T\mathbb{P}^1}$ allowed us to construct the uniton bundle from the uniton and the extra structure (reality, time invariance) fits naturally with this geometry. In this case, however, what appears natural is not entirely optimal. By transporting the uniton bundles to \mathbb{P}^2 via a birational equivalence, constructing and manipulating monads becomes much easier. To start with, the existence of monad representations is known.

Every operation we will make on \mathbb{P}^2 has its analogue on $\widetilde{T\mathbb{P}^1}$, but monads on projective spaces have two big advantages which we will exploit in interpreting the reality and triviality properties of the bundle. Namely,

- (1) \mathbb{P}^2 monads are self-dual, i.e. the transposed monad is a monad of the same form representing the dual bundle, and
- (2) we know when the bundle is trivial on hyperplanes. In the usual notation (described below), $\mathcal{V} \rightarrow \mathbb{P}^2$ is trivial on a hyperplane $L = \overline{p_1 p_2} \iff \det(K_{p_2} \circ J_{p_1}) \neq 0$, and

any choice of spanning representatives of $\ker K_{p_2}/\text{im } J_{p_1}$ frames $\mathcal{V}|_L$ canonically, in particular a basis for $\ker K_{p_1} \cap \ker K_{p_2}$ does. (See [OSS], [Do], [Hu].)

3.1 The birational equivalence. If X , Y and W are homogeneous coordinates on \mathbb{P}^2 , and λ and η base and fibre coordinates on $\widetilde{T\mathbb{P}^1}$ (see (2.1)),

$$(3.2) \quad \{X = \lambda Y, (X + Y)W = \eta Y^2\} \subset \widetilde{T\mathbb{P}^1} \times \mathbb{P}^2$$

is the graph of the birational equivalence which comes from

- (1) blowing up the point $(\lambda = -1, \eta = 0)$,
- (2) blowing down \widetilde{P}_{-1} (the proper transform of the fibre $\{\lambda = -1\}$), and
- (3) blowing down the image of G_∞ .

The birational equivalences we will need are given diagrammatically in figure 1.

Under the birational equivalence (3.2), the ruling $\{P_\lambda : \lambda \in \mathbb{P}^1\}$ of $\widetilde{T\mathbb{P}^1}$ is mapped to the pencil of lines $\{X = \lambda Y\}$ on \mathbb{P}^2 , except the fibre P_{-1} , which is mapped to a point. On \mathbb{P}^2 , the line $X + Y = 0$ is the exceptional divisor, and G_∞ becomes the point $[0, 0, 1] \in \mathbb{P}^2$. Since push forward gives an isomorphism ([An1, Lemma 7.2], [At]) between bundles on $\widetilde{T\mathbb{P}^1}$ and bundles on \mathbb{P}^2 , trivial on P_{-1} and $\{X + Y\}$ respectively, we will use the same letter for a bundle and its pullback.

Assume now that \mathcal{V} is a uniton bundle. Since \mathcal{V} is trivial on generic lines, \mathcal{V} admits a monad representation ([OSS, example 3, p249]), $\mathcal{V} = \ker K / \text{im } J$, where

$$(3.3) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^k \xrightarrow{J} \mathcal{O}_{\mathbb{P}^2}^{2k+N} \xrightarrow{K} \mathcal{O}_{\mathbb{P}^2}(1)^k \rightarrow 0$$

is a complex of linear maps, i.e. degree one homogeneous polynomials such that $K \circ J = 0$, on each fibre J is injective and K is surjective, and $k = c_2 \mathcal{V}$.

Since the birational equivalence blows down the fibre P_{-1} on which δ_t acts nontrivially and replaces it with an exceptional fibre on which δ_t must act discontinuously, one must either work with $\widetilde{T\mathbb{P}^1}$ monads or use another birational transformation, which we do in §8.

Clearly the monad representation (3.3) is not unique; $\text{Gl}(k) \times \text{Gl}(2k + N) \times \text{Gl}(k)$ acts on the vector spaces linearly inducing monad equivalences. Given that $\mathcal{V}|_{\{X+Y\}}$ is trivial, we can assume that the monad has the block form

$$(3.4) \quad \begin{aligned} J_W &= \begin{pmatrix} \mathbb{I} \\ 0 \\ 0 \end{pmatrix} & J_X - J_Y &= \begin{pmatrix} 0 \\ \mathbb{I} \\ 0 \end{pmatrix} & J_X + J_Y &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix} \\ K_W &= (0 \quad \mathbb{I} \quad 0) & K_X - K_Y &= (-\mathbb{I} \quad 0 \quad 0) & K_X + K_Y &= (-\alpha_2 \quad \alpha_1 \quad b) \end{aligned}$$

where the α_i , a and b are $k \times k$, $N \times k$ and $k \times N$ matrices, respectively. This form is stabilised by an action of $\text{Gl}(k) \times \text{Gl}(N)$. The $\text{Gl}(N)$ action corresponds to changes of frame. Since uniton bundles are framed it does not act.

The monad equation $K \circ J = 0$ is

$$(3.5) \quad [\alpha_1, \alpha_2] + ba = 0$$

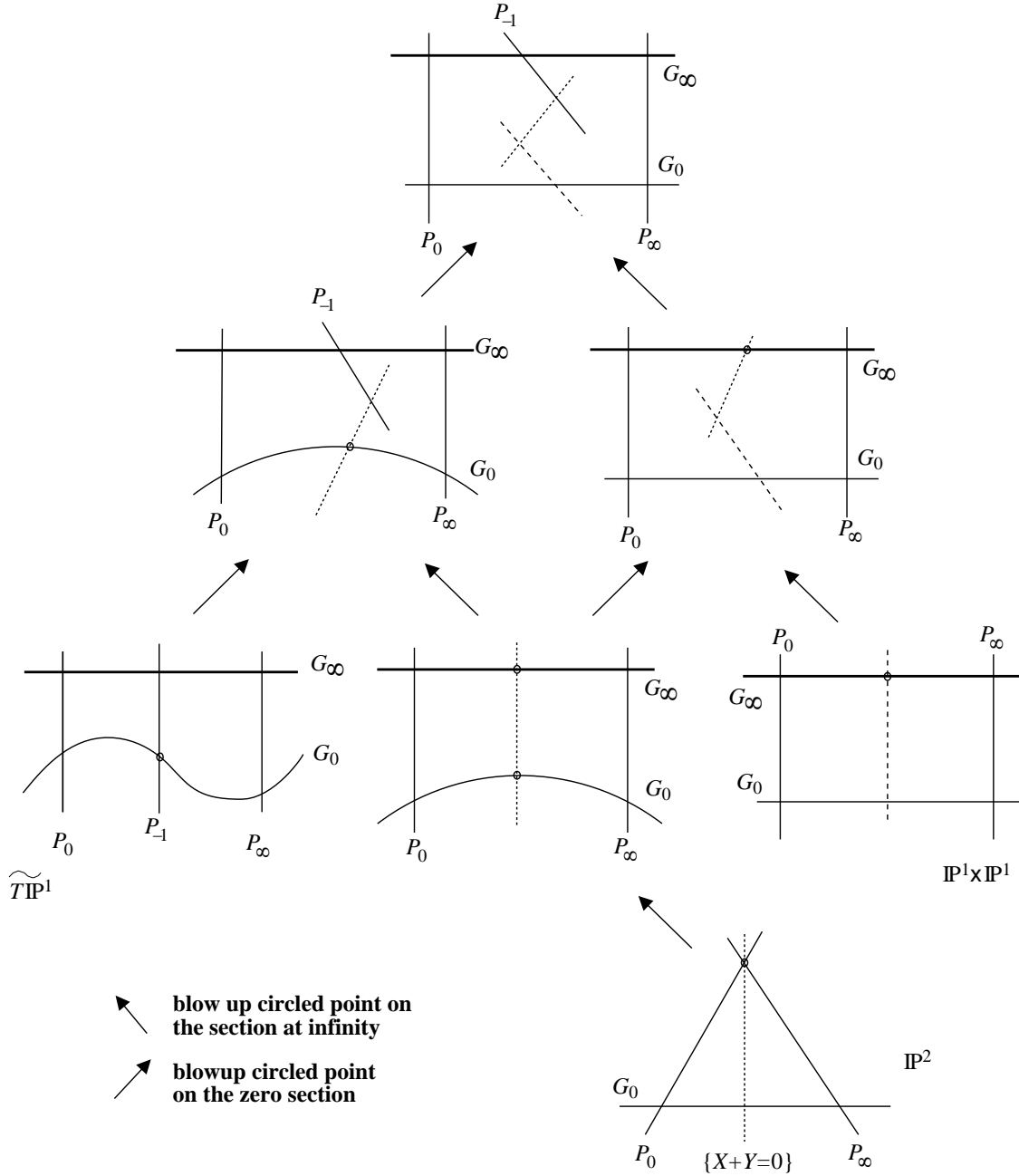


FIGURE 1. Birational equivalences of surfaces can be realised by sequences of blowings up and blowings down. Monad representations for generically trivial bundles exist for all rational surfaces and monad maps equating these representations are defined on a ‘common ancestor’ of the two surfaces. For example in the explicit monad construction of §5 we need to work on the topmost surface.

and nondegeneracy says that

$$\begin{pmatrix} \alpha_1 + X \\ \alpha_2 + Y \\ a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\alpha_2 - Y & \alpha_1 + X & b \end{pmatrix}$$

are respectively injective and surjective for all X, Y . Since

$$K_W \circ (\lambda J_X + J_Y) = \frac{1}{2}((1 + \lambda)\mathbb{I} - (1 - \lambda)\alpha_2),$$

$\mathcal{V}|_{\{X=\lambda Y\}}$ is holomorphically nontrivial iff $\frac{1+\lambda}{1-\lambda}$ is an eigenvalue of α_2 .

Since only $\mathcal{V}|_{\{X=0\}}$ and $\mathcal{V}|_{\{Y=0\}}$ are nontrivial,

$$(3.6) \quad \alpha_2 = \begin{pmatrix} -\mathbb{I} - 2\gamma' & \\ & \mathbb{I} + 2\gamma \end{pmatrix}$$

for some nilpotent matrices γ, γ' . (The twos will be convenient later.) If we correspondingly decompose

$$\alpha_1 = \begin{pmatrix} \alpha_{1,11} & \alpha_{1,12} \\ \alpha_{1,21} & \alpha_{1,22} \end{pmatrix}$$

the condition (3.5) implies that $\alpha_{1,12}$ and $\alpha_{1,21}$ are determined by γ, γ', a and b .

3.7 Reality. In interpreting reality, we are motivated by the geometry of the uniton bundle. The bundle \mathcal{V} is trivial on fibres other than P_0 and P_∞ and is specified by these jumping lines, i.e. by holomorphic bundles on neighbourhoods of P_0 and P_∞ with framings along G_∞ which tell how to glue them into the trivial bundle on $\widetilde{T\mathbb{P}^1} \setminus (P_0 \cup P_\infty)$.

Since the real structure fixes the framing of \mathcal{V} and exchanges P_0 and P_∞ , we know that $\mathcal{V}|_{P_0}$ and $\mathcal{V}|_{P_\infty}$ are ‘conjugate’. We can separate the bundle into two parts by gluing neighbourhoods of P_0 and P_∞ into separate trivial bundles, obtaining $\mathcal{V}_{\text{north}}$ and $\mathcal{V}_{\text{south}}$ say. Then

$$\mathcal{V}_{\text{north}} \cong \overline{\sigma^* \mathcal{V}_{\text{south}}^{\text{dual}}},$$

and the corresponding monads $(J', K'), (J'', K'')$ will satisfy $J'' = \overline{\sigma^* K'^t}, K'' = \overline{\sigma^* J'^t}$. Since

$$\begin{aligned} X &\mapsto \overline{Y} \\ \sigma^* : Y &\mapsto \overline{X} \\ W &\mapsto -\overline{W} \end{aligned}$$

we obtain (after putting the monad back into normal form)

$$\begin{aligned} \alpha_1'' &= \alpha_1'^* \\ \alpha_2'' &= -\alpha_2'^*, \text{ i.e. } \gamma = \gamma'^* \\ a'' &= b'^* \\ b'' &= a'^*, \end{aligned}$$

in particular α'_i and α''_i have the same size.

Finally, the monad equation determines $\alpha_{1,12}$ and $\alpha_{1,21}$ to be

$$(3.8) \quad \begin{aligned} \alpha_{2,12} &= -\frac{1}{2} \sum \gamma^{*(i-1)} (\mathbb{I} + \gamma^*)^{-i} a'^* a' (\mathbb{I} + \gamma)^{-i} \gamma^{i-1} \\ \alpha_{2,21} &= \frac{1}{2} \sum \gamma^{i-1} (\mathbb{I} + \gamma)^{-i} b' b'^* (\mathbb{I} + \gamma^*)^{-i} \gamma^{*(i-1)}. \end{aligned}$$

Summing up, a (real) uniton bundle is represented by a \mathbb{P}^2 monad

$$\begin{aligned} \alpha_2 &= \begin{pmatrix} -\mathbb{I} - 2\gamma^* & \\ & \mathbb{I} + 2\gamma \end{pmatrix} \\ \alpha_1 &= \begin{pmatrix} \alpha_1'^* & \sum \cdots \\ \sum \cdots & \alpha_1' \end{pmatrix} \\ b &= \begin{pmatrix} a'^* \\ b' \end{pmatrix} \\ a &= (b'^* \quad a'). \end{aligned}$$

3.9 Real Triviality. Real sections of $\widetilde{T\mathbb{P}^1}$ are smooth, linearly-equivalent curves of self-intersection two. By blowing up and down twice, some of these curves become singular, which seems unpleasant, but look again at the singular sections.

Sections $\eta = a + b\lambda + c$ of $\widetilde{T\mathbb{P}^1}$ are parametrised by \mathbb{C}^3 . The sections $G_{(z,t,\bar{z})} = \{\eta = \frac{1}{2}(z - 2t\lambda - \bar{z}\lambda^2)\}$ for $t, z \in \mathbb{C}$ are time translates of real sections. Uniton bundles are required to be trivial on real sections. Since they are invariant under time translation, it is enough to know that for each $z \in \mathbb{C}$, \mathcal{V} is trivial on $G_{(z,t,\bar{z})}$ for some $t \in \mathbb{C}$. The sections with $t = (\bar{z} - z)/2$ all contain the point $(\lambda = -1, \eta = 0)$. So the proper transforms of these sections are singular (the union of two curves). Since they still have self-intersection two, and are effective, they are unions of two hyperplanes $(\bar{z}X - zY + 2W)(X + Y) = 0$, one of which is the exceptional fibre. So $\mathcal{V}|_{G_{(z,t,\bar{z})}}$ is trivial iff \mathcal{V} is trivial on the hyperplane $\bar{z}X - zY + 2W = 0$. So \mathcal{V} is trivial on all real sections iff

$$(3.10) \quad \begin{aligned} 0 &\neq \det \left(\frac{1}{2}(z - \bar{z})K_W + K_X + K_Y \right) \left(\frac{1}{2}(-z - \bar{z})J_W + J_X - J_Y \right) \\ &= \det \left(\frac{1}{2}(z - \bar{z})\alpha_2 + \frac{1}{2}(z + \bar{z})\mathbb{I} - \alpha_1 \right) \end{aligned}$$

for all $z \in \mathbb{C}$. In §7 we will show that real triviality is implied by reality, but it will be helpful to have the expression (3.10) in the next section.

4. FORMULA

4.1 Formula. Now we are in a position to use the monodromy construction of the uniton (2.12), to get a formula for the uniton in terms of the monad data α_1, α_2, a, b . It suffices

to parametrise the sections of $\mathcal{V}|_{\{X=\lambda Y\}}$ and of $\mathcal{V}|_{\{\bar{z}X - zY + 2W=0\}}$ because the section at infinity is collapsed to a point, and both the bundle and corresponding uniton are time-translation invariant.

Sections of $\mathcal{V}|_{\{\bar{z}X - zY + 2W=0\}}$ are parametrised by

$$\ker K_{p_1} \cap \ker K_{p_2}$$

where $\overline{p_1 p_2} = \{\bar{z}X - zY + 2W = 0\}$. Taking $p_1 = [2, 2, z - \bar{z}]$ and $p_2 = [2, -2, -z - \bar{z}]$,

$$H^0(\mathcal{V}|_{\overline{p_1 p_2}}) \cong \ker \begin{pmatrix} K_{p_2} \\ K_{p_1} \end{pmatrix} = \ker \begin{pmatrix} -\mathbb{I} & \frac{1}{2}(z - \bar{z})\mathbb{I} & 0 \\ -\alpha_2 & \alpha_1 - \frac{1}{2}(z + \bar{z})\mathbb{I} & b \end{pmatrix}.$$

Real triviality (3.10) says

$$\det \begin{pmatrix} -\mathbb{I} & iy\mathbb{I} \\ -\alpha_2 & \alpha_1 - x \end{pmatrix} = (-1)^{k/2} \det(\alpha_1 - (x + iy\alpha_2)) \neq 0$$

which implies that the kernel is

$$(4.2) \quad \left\{ \begin{pmatrix} -iy(\alpha_1 - (x + iy\alpha_2))^{-1}bs \\ -(\alpha_1 - (x + iy\alpha_2))^{-1}bs \\ s \end{pmatrix} \in \mathbb{C}^{2k+N} : s \in \mathbb{C}^N \right\}.$$

Sections of $\mathcal{V}|_{\{X=\lambda Y\}}$ are similarly given by

$$(4.3) \quad \begin{pmatrix} (\alpha_2 + \frac{\lambda+1}{\lambda-1}\mathbb{I})^{-1}bs \\ 0 \\ s \end{pmatrix} \in \mathbb{C}^{2k+N}$$

as s varies in \mathbb{C}^N .

The evaluation map $H^0(\{X = \lambda Y\}, \mathcal{V}) \rightarrow \mathcal{V}_p$ is given by (4.3) \mapsto (4.3) + im J_p . To compute

$$H^0(\{\bar{z}X - zY + 2W = 0\}, \mathcal{V}) \rightarrow \mathcal{V}_p \rightarrow H^0(\{X + \lambda Y\}, \mathcal{V})$$

we have to take the expression (4.2) and add an element of im J_p to obtain a representative of the form (4.3).

At the point of intersection $p = [-\lambda, -1, \bar{z}\lambda - z]$

$$\frac{2}{1-\lambda} J_p = \begin{pmatrix} \alpha_1 + \frac{z-\lambda\bar{z}}{\lambda-1} \\ \alpha_2 + \frac{\lambda+1}{\lambda-1} \\ a \end{pmatrix}.$$

When $\lambda \in \mathbb{C}^*$, we can translate (4.2) into the form of (4.3) by adding

$$\frac{2}{1-\lambda} J_p \left((\alpha_2 + \frac{\lambda+1}{\lambda-1})^{-1} (\alpha_1 - (x + iy\alpha_2))^{-1} bs \right)$$

giving

$$\begin{pmatrix} \left(-iy + \left(\alpha_1 + \frac{z - \lambda \bar{z}}{\lambda - 1} \right) \right) \left(\alpha_2 + \frac{\lambda + 1}{\lambda - 1} \right)^{-1} (\alpha_1 - (x + iy\alpha_2))^{-1} bs & \\ 0 & \\ \left(\mathbb{I} + a \left(\alpha_2 + \frac{\lambda + 1}{\lambda - 1} \right)^{-1} (\alpha_1 - (x + iy\alpha_2))^{-1} b \right) s & \end{pmatrix}.$$

Note that the resulting map $\mathbb{C}^N \rightarrow \mathbb{C}^N$ is linear. Since the parametrisations of $H^0(\{X = \lambda Y\}, \mathcal{V})$ agree at their point of intersection $[0, 0, 1]$, we can compute the monodromy construction of the extended solution as a composition of two linear maps—a matrix:

$$(4.4) \quad E_\lambda = \left(\mathbb{I} + a \left(\alpha_2 + \frac{\lambda + 1}{\lambda - 1} \right)^{-1} (\alpha_1 - (x + iy\alpha_2))^{-1} b \right)^{-1} \begin{pmatrix} \\ \\ \left(\mathbb{I} + a \alpha_2^{-1} (\alpha_1 - (x + iy\alpha_2))^{-1} b \right) \end{pmatrix};$$

when $\lambda = 1$, $S = E_1$ is the sought expression of Theorem A.

Remark 4.5. We can verify directly that these solutions are unitary using the reality conditions. From reality,

$$\begin{aligned} S^* &= \mathbb{I} + b^* (\alpha_1^* - (x - iy\alpha_2^*))^{-1} \alpha_2^{*-1} a^* \\ &= \mathbb{I} - a \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \left(\begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \alpha_1 \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \right. \\ &\quad \left. - \left(x + iy \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \alpha_2 \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \right) \right)^{-1} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \alpha_2^{-1} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} b \\ &= \mathbb{I} - a (\alpha_1 - (x + iy\alpha_2))^{-1} \alpha_2^{-1} b. \end{aligned}$$

Let

$$\begin{aligned} X_1 &= (\alpha_1 \alpha_2 - x \alpha_2 - iy \alpha_2^2) \\ X_2 &= (\alpha_2 \alpha_1 - x \alpha_2 - iy \alpha_2^2), \end{aligned}$$

the monad equation $[\alpha_1, \alpha_2] + ba = 0$ is equivalent to $X_1 - X_2 = -ba$. Using this notation,

$$\begin{aligned} S^* S &= \mathbb{I} + a(X_1^{-1} - X_2^{-1})b - aX_2^{-1}baX_1^{-1}b \\ &= \mathbb{I} + a(X_1^{-1} - X_2^{-1})b - aX_2^{-1}(X_2 - X_1)X_1^{-1}b \\ &= \mathbb{I}. \end{aligned}$$

Remark 4.6. We can simplify the uniton equation as well. But we can only show that it is satisfied when $\gamma = 0$. Now let $X = \alpha_1 - (x + iy\alpha_2)$, so

$$\begin{aligned} S &= \mathbb{I} + a \alpha_2^{-1} X^{-1} b \\ S^{-1} &= S^* = \mathbb{I} - a X^{-1} \alpha_2^{-1} b. \end{aligned}$$

We calculate

$$\begin{aligned}
\frac{\partial}{\partial x} S &= a\alpha_2^{-1}X^{-2}b \\
S^* \frac{\partial}{\partial x} S &= aX^{-1}\alpha_2^{-1}X^{-1}b \\
\frac{\partial}{\partial x} (S^* \frac{\partial}{\partial x} S) &= aX^{-2}\alpha_2^{-1}X^{-1}b + aX^{-1}\alpha_2^{-1}X^{-2}b \\
\frac{\partial}{\partial y} S &= ia\alpha_2^{-1}X^{-1}\alpha_2X^{-1}b \\
S^* \frac{\partial}{\partial y} S &= iaX^{-2}b \\
\frac{\partial}{\partial y} (S^* \frac{\partial}{\partial y} S) &= -aX^{-1}\alpha_2X^{-2}b - aX^{-2}\alpha_2X^{-1}b.
\end{aligned}$$

So

$$\text{uniton equation} = aX^{-2} \{ \alpha_2^{-1}X + X\alpha_2^{-1} - \alpha_2X - X\alpha_2 \} X^{-2}b.$$

When $\gamma = 0$, so that α_2 is diagonal, the middle factor is zero. But it is not zero in general (use the example of a $U(3)$ monad in section 5).

5. EXAMPLE OF A $U(3)$ UNITON

We have two ways of describing holomorphic vector bundles, in terms of clutching functions or in terms of monad data. The monad approach should be thought of as an attempt to reduce to the case we understand, line bundles, which form the commutative part of the category of bundles. Our tool is the resolution and the homological algebra surrounding it. As we have seen, it is an effective tool, but not without its drawbacks. On a practical level, we know that time invariance is effectively a restriction on the bundle and not an extra structure. This was clear and testable in terms of the clutching data in the example in [An1], but not obvious for monads.

Ideally, one would construct a dictionary to translate from monad to clutching data and back and take advantage of both representations. This was one motivation for trying to directly construct the monad data corresponding to the clutching data in [An1]. We were able to do this for this example, but it's not clear how to do it in general. The procedure is as follows.

Beilinson's theorem on the existence of a monad representative for $\mathcal{V} \mapsto X$ is proven by defining a resolution of $\pi_1^* \mathcal{V} \otimes \mathcal{O}_\Delta$ on $X \times X$ whose cohomology is the bundle to be represented. We can push the resolution down to X via two projections. The first push-down gives back \mathcal{V} . In good cases, the spectral sequence for the other direct image doesn't degenerate to the bundle, but has a nontrivial E_2 term whose entries are twisted cohomology groups of \mathcal{V} which form the monad (see [OSS]). It's not surprising that this can be done for generically trivial bundles on rational surfaces, since such bundles can be pulled back to \mathbb{P}^2 via generic birational equivalences. The author's thesis contains the details of this for $\widehat{T\mathbb{P}^1}$. We omit the justification as we only need it for this example.

To make the computations easier, cut out a neighbourhood of $\{\lambda = 0\}$ in $\widetilde{T\mathbb{P}^1}$ and glue it into the trivial bundle on $(x, z) \in \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{Z}(p, q) = \mathcal{Z} \otimes \mathcal{O}(\pi_1^{-1}(p \text{ points}) + \pi_2^{-1}(q \text{ points}))$. The Leray spectral sequence for $H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{V}(p, q))$ gives ‘natural’ bases for Čech cohomology. Computing the direct image of the maps in the resolution (i.e. their natural transforms) means writing $H^1(x)$, etc, in terms of the chosen bases. The direct image of $1 \otimes x_2 - x_1 \otimes 1$ is then $(H^1(x) - xH^1(1))$, where tensor product and addition of maps become matrix multiplication and addition. The resulting $\mathbb{P}^1 \times \mathbb{P}^1$ monad (for the jump at P_0 only) is

$$\mathcal{O}(-1, -2)^4 \xrightarrow{\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ -1 & 0 & x & 0 \\ 0 & -1 & 0 & x \end{pmatrix}} \mathcal{O}(-1, -1)^4 \oplus \mathcal{O}(0, 2)^7 \xrightarrow{\begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & -z/2 & -1/2 \\ -z & 1 & 0 & 0 \end{pmatrix}} \mathcal{O}(0, -1)^4.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -z & 0 & 0 \\ z & 0 & 0 & 0 \\ z & -1 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & z/2 & 1/2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

We want a monad on \mathbb{P}^2 , so make the substitutions

$$\begin{aligned} x &= X/Y \\ z &= WY/(X + Y)^2 \end{aligned}$$

coming from the birational equivalence in figure 1. (This is well-defined on the blown-up variety at the top of figure 1.) Before we blow down, we need to make a monad transformation given by

$$(1 + \lambda)^2 \mathbb{I}_4 \quad \begin{pmatrix} (1 + \lambda)^2 \mathbb{I}_4 & \\ & \mathbb{I}_7 \end{pmatrix} \quad \mathbb{I}_4$$

(which again makes sense on the double blow up). We then use the group of isomorphisms of the \mathbb{P}^2 monad to put the resulting monad into normal form yielding normalised data

$$\begin{aligned} \gamma &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \alpha'_1 &= \begin{pmatrix} 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ a' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix} & b' &= \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Plugging this into the formula in Theorem A, we get the same solution we obtained in [An1].

6. INTERPRETATION OF UNITON NUMBER

We will now show that unton number corresponds to the length of the polar jumping line. Recall that $S \in \text{Harm}(\mathbb{S}^2, \text{U}(N))$ has unton number n if S admits an extended solution of the form

$$\tilde{E}_\lambda = T_0 + \lambda T_1 + \dots \lambda^n T_n$$

and this is the shortest possible such solution. We will assume, without loss of generality, that \tilde{E}_λ is in Uhlenbeck normal form, i.e.

$$\text{span}\{\text{im } T_0(z) : z \in \mathbb{C}\} = \mathbb{C}^N.$$

For our purposes, the best definition of the length of $\mathcal{V}|_{P_0}$ is as the largest l such that there exists a map

$$\mathcal{O}_{P_0^{(l)}}(1) \xrightarrow{s} \mathcal{V}_{P_0^{(l)}}, \quad s|_{P_0} \neq 0,$$

equivalently

$$\mathcal{V}_{P_0^{(l)}} \xrightarrow{s} \mathcal{O}_{P_0^{(l)}}(-1), \quad s|_{P_0} \neq 0.$$

In terms of our monad, this is the smallest l such that $\gamma^{l+1} = 0$.

Proof. The first characterisation follows from the observation that if p_1 and p_2 are two points on $P_0 \subset \mathbb{P}^2$, then

$$\ker K_{p_2} \circ J_{p_1} = \{s \in H^0(P_0, \mathcal{V}) : s(p_1) = 0\} \cong H^0(P_0, \mathcal{V}(-1)),$$

and that this works just as well for $P_0^{(l)}$ in which case all objects are defined over $\mathbb{C}[\lambda]/(\lambda^{n+1})$ instead of over \mathbb{C} . Since the dual bundle is described by the transposed monad, this gives the equivalence. See [Ti] for another definition of length.

The bundle \mathcal{V} comes with two sorts of framings. The first is a framing of \mathcal{V} over G_∞ and $\widetilde{T\mathbb{C}^*}$ (the union of the nonpolar fibres), call it g . The other is a family of framings f_y over G_y where y is in a neighbourhood of the real sections in \mathbb{C}^3 . They are related by the change of frame $E_\lambda(y)$

$$g = E_\lambda(y) \cdot f_y$$

(which is only defined over $\{\lambda \in \mathbb{C}^*\} \cap G_y$).

Among extended solutions, E_λ is determined by the property $E_\lambda(\infty) = \mathbb{I}$. This implies

$$E_\lambda(y) = \tilde{E}_\lambda(\infty)^{-1} \tilde{E}_\lambda(y).$$

In particular, the sections

$$\tilde{E}_\lambda(y) \cdot f_y = \tilde{E}_\lambda(\infty) \cdot g$$

are holomorphic on a neighbourhood of P_0 in $\widetilde{T\mathbb{P}^1}$ and have full rank away from P_0 .

Since E_λ is an analytic function on \mathbb{S}^2 , it extends to a holomorphic function on an open subset of \mathbb{C}^2 , which we extend trivially to \mathbb{C}^3 . This is what we mean by $E_\lambda(y)$. By G_z we will mean $G_{(z,0,\bar{z})}$.

Since \tilde{E}_λ is in Uhlenbeck normal form, we can find z_0, z_1 such that

$$T_n(z_0)^* T_0(z_1) \neq 0.$$

The reality condition on E_λ

$$E_\lambda^{-1} = (E_{\bar{\lambda}^{-1}})^* = \lambda^{-n} T_n^* + \cdots + \lambda^{-1} T_1^* + T_0^*$$

implies that the N sections

$$\lambda^n \tilde{E}_\lambda(z_0)^{-1} \tilde{E}_\lambda(y) \cdot f_y = \lambda^n \tilde{E}_\lambda(z_0)^{-1} \tilde{E}_\lambda(\infty) \cdot g$$

have zeros along $P_0^{(n-1)} \cap G_{z_0}$, but are not all zero on $P_0 \cap G_{z_1}$. This shows that $l \geq n-1$ (by the first characterisation of length).

Assume, on the other hand, that $l > n-1$ and let s be a section

$$\mathcal{V} \xrightarrow{s} \mathcal{O}_{P_0^{(l)}}(-1); \quad s|_{P_0} \neq 0.$$

We can find a $y_0 \in \mathbb{R}^3$ and a $v \in \mathcal{V}_{P_0 \cap G_{y_0}}$ such that $s(v) \neq 0$. It follows that the sections

$$\lambda^n E_\lambda(y_0)^{-1} E_\lambda(y) \cdot f_y$$

on $P_0^{(n)} \cap G_{y_0}$ are not all mapped to zero by s , but since their image is a section of $\mathcal{O}_{P_0^{(n)}}(-1) \cong \mathcal{O}_{P_0}(-1)^{\oplus(n+1)}$, they must be zero. We conclude that $n = l+1$.

7. REAL TRIVIALITY

A bundle over \mathbb{P}^1 with zero first Chern class is either trivial or contains a positive subbundle. To show $\mathcal{V}|_{G_{\text{real}}}$ is trivial we will show $H^0(G_{\text{real}}, \mathcal{V})$ contains no sections with isolated zeros.

Recall that \mathcal{V} is trivial on all real sections iff it is trivial on the σ -invariant lines $\{\bar{z}X - zY + 2W = 0\}$. Let \overline{pq} be such a line, with $\sigma(p) \neq q$.

We need the

Lemma 7.1. *There exist hermitian metrics on \mathbb{C}^k and \mathbb{C}^{2k+N} such that*

$$(J_{\sigma(p)})^* = K_p.$$

This allows us to calculate

$$\begin{aligned} H^0(\overline{pq}, \mathcal{V}) &\cong \ker K_p \cap \ker K_q \\ &= (\text{im } K_p^*)^\perp \cap \ker K_q \\ &= (\text{im } J_{\sigma(p)})^\perp \cap \ker K_q \end{aligned}$$

which implies

$$\begin{aligned} \{s \in H^0(\overline{pq}, \mathcal{V}) : s(\sigma(p)) = 0\} &\cong \ker K_p \cap \ker K_q \cap \operatorname{im} J_{\sigma(p)} \\ &= \{0\}. \end{aligned}$$

It follows that $\mathcal{V}|_{\overline{pq}}$ is trivial.

Proof of Lemma. Assume J and K are in the real form specified in Theorem B. Let

$$\Omega \stackrel{\text{def}}{=} \begin{pmatrix} & \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \\ \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} & \mathbb{I} \end{pmatrix} \quad \text{and} \quad \omega \stackrel{\text{def}}{=} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix},$$

and define hermitian forms on \mathbb{C}^{2k+N} and \mathbb{C}^k by

$$\langle v_1, v_2 \rangle \stackrel{\text{def}}{=} \overline{v_2}^t \Omega v_1 \quad \text{and} \quad \langle u_1, u_2 \rangle \stackrel{\text{def}}{=} \overline{u_2}^t \omega u_1.$$

Since

$$\begin{aligned} \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \alpha_2 \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} &= -\overline{\alpha_2}^t & \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} b &= \bar{a}^t \\ \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} \alpha_1 \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} &= \overline{\alpha_1}^t & a \begin{pmatrix} & \mathbb{I} \\ \mathbb{I} & \end{pmatrix} &= \bar{b}^t \end{aligned}$$

$\overline{\sigma(X+Y)} = X+Y$, $\overline{\sigma(X-Y)} = -(X-Y)$ and $\overline{\sigma(W)} = -W$, we have

$$K_p = \overline{\Omega J_{\sigma(p)} \omega}^t = (J_{\sigma(p)})^*$$

as required.

8. TIME TRANSLATION

Because

- (1) $\mathcal{V}|_{\text{nonpolar fibres}}$ is holomorphically trivial,
- (2) $\delta_t : \widetilde{T\mathbb{P}^1} \rightarrow \widetilde{T\mathbb{P}^1}$ preserves the fibres,
- (3) δ_t fixes G_∞ and its lift $\tilde{\delta}_t$ fixes $\mathcal{V}|_{G_\infty}$, and
- (4) the union of the nonpolar fibres is a Zariski open set,

the lift of time translation to the bundle is unique if it exists. It therefore makes sense to refer to ‘time-invariant’ holomorphic bundles. We must identify these bundles.

Since the real structure, $\tilde{\sigma}$, mirrors the holomorphic structure at P_0 and P_∞ , the same argument shows that time invariance is a local property, depending on the framed holomorphic structure in a neighbourhood of either pole. In terms of monads, we may work with half the monad (as in §3.7) and look for conditions under which time translation exists as a monad isomorphism.

In section 3 we fixed a birational equivalence of $\widetilde{T\mathbb{P}^1}$ and \mathbb{P}^2 which was convenient for deriving the formula. In fact, any equivalence by which (complete) uniton bundles could be pushed forward must send time translation to a discontinuous group of transformations on \mathbb{P}^2 . Working with half the monad (and half the bundle, $\mathcal{V}_{\text{north}}$ of §3.7), we may blow up $\hat{\eta} = 0, \lambda = \infty$, instead. The resulting equivalence,

$$Y\lambda = X, \quad Y\eta = W,$$

sends $\delta_t : \lambda \mapsto \lambda, \eta \mapsto \eta + \lambda t$ to a *linear* transformation of \mathbb{P}^2 :

$$(8.1) \quad X \mapsto X, \quad Y \mapsto Y \text{ and } W \mapsto W + Yt.$$

Since time invariance is a local property, it doesn't matter which birational equivalence we choose as we get equivalent monads. So we may as well assume we are working with the $1/2$ monad $(\alpha'_1, \alpha'_2, a', b')$. We are looking for a map $\mathcal{V} \rightarrow \delta_t^* \mathcal{V}$. Such a map exists iff the respective monads are equivalent under the group action. Pulling J and K back by δ_t (8.1) disturbs the normalisation we have chosen. Multiplying $\mathcal{O}^{\oplus(2k+N)}$ by

$$\begin{pmatrix} \mathbb{I} & t\mathbb{I} & \\ & \mathbb{I} & \\ & & \mathbb{I} \end{pmatrix}$$

restores the normalisation, but sends

$$\begin{aligned} \alpha'_1 &\mapsto \alpha'_1 - 2t\gamma \\ \alpha'_2 &\mapsto \alpha'_2 \\ a' &\mapsto a' \\ b' &\mapsto b'. \end{aligned}$$

The two are equivalent iff there exists a one parameter subgroup $G(t) \in \text{Gl}(k/2)$ which fixes γ, a' and b' , and sends α'_1 to $\alpha'_1 - 2t\gamma$. Infinitesimally, this says there exists $g \in \mathfrak{gl}(k/2)$ such that

$$(8.2) \quad [g, \alpha'_1] = \gamma, \quad [g, \gamma] = 0, \quad gb' = 0, \quad a'g = 0.$$

As we have said, g is unique up to the group action. When it exists, it is the δ of Theorem B.

Lemma 8.3. *No monad with*

$$\gamma \in \left\{ (N_n), \begin{pmatrix} N_2 & \\ & 0 \end{pmatrix}, \begin{pmatrix} N_m & \\ & Z_r \end{pmatrix} : n > 1, m > 2, r > 0 \right\}$$

where

$$N_n \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathfrak{gl}(n), \text{ and } Z_r \stackrel{\text{def}}{=} (0) \in \mathfrak{gl}(r)$$

exists.

Proof. Let $\alpha = \alpha'_1$, γ , $a = a'$, $b = b'$ be monad data for $\mathcal{V}_{\text{north}}$, and let δ and g represent the infinitesimal time translation and group action respectively. Assume γ of the form above. We first show that $n \leq 2$.

If $[g, \gamma] = 0$, then

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where $g_{11} \in \mathfrak{gl}(n)$ is upper-diagonal Toeplitz, g_{12} is zero except in its first row and g_{21} is zero except in its last column. The same applies to δ .

Case I. If $a_1 \neq 0$ (the first column), we can assume (use the group action to insure) that $(a_1, a_j) = 0$, for all $j > 1$. With this normalisation $ag = 0$ implies $g_{11} = 0$, $g_{12} = 0$. These restrictions apply as well to δ . The condition $[\alpha, \delta] = \gamma$ now implies $n \leq 2$, and when $n = 2$, that the last column of δ_{21} is not zero, while the second row of α is zero. Nondegeneracy then implies that the second row of b is not zero, but $\delta b = 0$ then implies $\delta = 0$, which contradicts the assumption that $[\alpha, \delta] = \gamma$.

Case II. Assume that the first $n_a \geq 1$ columns of a are zero, as are the first n_g columns of g . Then $ag = 0$ implies $n_a + n_g \geq n$.

If $n_a \geq 2$, $[\alpha, \gamma] = ba$ implies that the first column of α must be zero, violating nondegeneracy. If, on the other hand, $n_g \geq n - 1$, $[\alpha, \delta] = \gamma$ is only possible when $n \leq 2$.

We are left with the case $n = 2$, $a_1 = 0$, and $b_2 = 0$ (by symmetry). The monad equation and nondegeneracy imply

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \neq 0 \\ \alpha_{31} \neq 0 & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \neq 0 \\ 0 \\ b_3 \neq 0 \end{pmatrix},$$

$$a = (0 \quad a_2 \neq 0 \quad a_3 \neq 0),$$

and $a\delta = 0$, $\delta b = 0$ imply

$$\delta = \begin{pmatrix} 0 & \delta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$([\alpha, \delta])_{32} = \gamma_{32} = 0$ implies $\delta = 0$ which contradicts $[\alpha, \delta] = \gamma$. \square

It is possible to extend this argument, but not (it seems) without degenerating into a comedy of cases. This much implies

Corollary 8.4. *Two-unitons have normalised energy at least four. This bound is sharp (see §5).*

9. MONADS WITH $\gamma = 0$

In the $\gamma = 0$ case, we have just seen that time invariance poses no additional constraints on the monad. We will now use the residual action of $G \in \mathrm{Gl}(k/2)$ and the nondegeneracy condition to put α'_1 , a' and b' into a normal form. In the next section we will then specialise to the well-known $\mathrm{U}(2)$ unitons and show how the monad data is equivalent to the pole and principal part description of based rational maps.

Again, we choose to work with half the monad. Since $\mathrm{Gl}(k/2)$ acts on α'_1 by conjugation, we can put α'_1 into Jordan normal form. If we agree to a lexicographical ordering of \mathbb{C} , we can fix the order of the Jordan blocks up to a permutation of blocks with the same rank and eigenvalue. Unless α'_1 is diagonalisable with distinct eigenvalues, its Jordan form has a nonzero stabiliser which continues to act on a' and b' . Once we divide α'_1 into groups of Jordan blocks, each group with a distinct eigenvalue and rank, we get a corresponding decomposition of $\mathbb{C}^{k/2}$ into α'_1 -invariant subspaces. They are the invariant subspaces of the stabiliser (under the standard action of $\mathrm{Gl}(k/2)$ on $\mathbb{C}^{k/2}$), so we may consider them one at a time.

Assume for the moment that α'_1 has exactly m Jordan blocks of size l . It is convenient to assume

$$\alpha'_1 = \begin{pmatrix} \nu & & & 1 & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & \\ & & & \ddots & & 1 \\ & & & & \ddots & \\ & & & & & \nu \end{pmatrix},$$

with lm ν 's and $m(l-1)$ 1's, because the stabiliser is then in block form:

$$\mathrm{Stab}_{\alpha'_1} \mathrm{Gl}(k/2) = \left\{ \begin{pmatrix} g_0 & g_1 & \cdots & g_{l-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & g_1 \\ & & & g_0 \end{pmatrix} : \begin{array}{l} g_0 \in \mathrm{Gl}(m) \\ g_i \in \mathfrak{gl}(m) \\ i = 1, \dots, l-1 \end{array} \right\}.$$

The injectivity of J imposes an independence condition on a' . At $\lambda = 0, \eta = -\nu$, $\mathbb{I}\eta + \alpha'_1$ is singular; the first l columns are zero, so if $u \in \mathbb{C}^m$

$$\begin{pmatrix} u \\ 0 \end{pmatrix} \in \begin{pmatrix} \mathbb{C}^m \\ \oplus \\ \mathbb{C}^{m(l-1)} \end{pmatrix} \xrightarrow{J|_{(\lambda, \eta) = (0, -\nu)}} \begin{pmatrix} (0) \\ a' \begin{pmatrix} u \\ 0 \end{pmatrix} \end{pmatrix} \in \begin{pmatrix} \mathbb{C}^{2ml} \\ \oplus \\ \mathbb{C}^N \end{pmatrix}.$$

Since J is injective, the first m columns of a' must be independent. Similarly, the surjectivity of K at the same point implies that the last m rows of b' are independent.

Since g_0 acts on $(a')_1, \dots, (a')_m$ by right multiplication, a' (and b') define points on a Grassmannian:

$$\{m \text{ independent } N\text{-vectors}\} / \mathrm{Gl}(m) \cong \mathrm{Gr}_m(\mathbb{C}^N).$$

If α'_1 has Jordan blocks of different rank but the same eigenvalue, we get an element of a Stiefel manifold, a space of linearly independent subspaces with prescribed ranks.

In any case, within each invariant block, we can use the action of g_0 to put the last m rows of b' into some normal form, say the one which gives coordinates on the Schubert cycles [GH,1.5]. This reduces the stabiliser to the subgroup $\{(\mathbb{I}, g_1, \dots, g_{l-1})\}$.

There are two ‘standard’ metrics on \mathbb{C}^N each of which can be used to put b' into a normal form. If we put the hermitian metric on \mathbb{C}^N , we can use the remaining action of the stabiliser to put the first $m(l-1)$ columns of b' perpendicular to the rest

$$\{b'_1, b'_2, \dots, b'_{m(l-1)}\} \subset \{b'_{m(l-1)+1}, b'_{m(l-1)+2}, \dots, b'_{ml}\}^{\perp_{\text{hermitian}}}.$$

This shows that the data b' describe a point in the orthogonal bundle to the universal bundle contained in $\underline{\mathbb{C}}_{Gr_m(\mathbb{C}^N)}^N$.

Alternatively, we can put the holomorphic Euclidean metric on \mathbb{C}^N . In this case we have to worry about null vectors, so the same procedure doesn’t work. Instead we have to consider the usual coordinate patches of $Gr_A(\mathbb{C}^N)$. If the last rows of b' are in Schubert cycle form, i.e.

$$\begin{array}{cccccccccccccccc} & & & i_1 & & & & i_2 & & & & i_3 & & \dots \\ & & & \downarrow & & & & \downarrow & & & & \downarrow & & \\ \begin{pmatrix} 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \end{array}$$

then we can make the first $m(l-1)$ rows of b' perpendicular to $\{e_{i_1}, e_{i_2}, \dots, e_{i_m}\}$ (in either metric).

Apply this procedure to each invariant subspace of $\text{Stab}_{\alpha'_1} \text{Gl}(k/2)$ and we are left with a normalised monad which uniquely represents the bundle E , i.e. its stabiliser in the group of monad isomorphisms is the trivial subgroup.

Remark 9.1. The reason for apparently two methods of normalising is simple. The moduli space of allowed a' ’s given a choice of α'_1 is a bundle over the appropriate Grassmannian or Stiefel manifold. As complex bundles, they are trivial and admit global frames. Holomorphically, however, they are not and we have to make different normalisations above the various Schubert cells.

Remark 9.2. To see directly that $\gamma = 0$ monads give 1-unitons, we can simplify the formula in Theorem A to something which looks like the holomorphic map composed with the Cartan embedding. We could then imitate the extended solution given in [Uhl].

10. EXAMPLE: $\text{Harm}(\mathbb{S}^2, \text{U}(2))$

This section treats the simplest case: $\text{U}(2)$ unitons given by rank-two uniton bundles. It demonstrates the formula and points out an important distinction between maps into

symmetric spaces and maps into $U(N)$. Inside $U(N)$, holomorphic and antiholomorphic maps into symmetric spaces look the same.

It is well known that harmonic maps $\mathbb{S}^2 \rightarrow U(2)$ factor through spheres and hence are closely linked to rational maps. We will give an ahistorical proof that based unitons correspond to rational maps ($\mathbb{P}^1 \rightarrow \mathbb{P}^1$), and show that the action of $U(2)$ on $\text{Harm}^*(\mathbb{S}^2, U(2))$ by conjugation ($S \mapsto USU^*$) corresponds to the usual $\text{Gl}(2)$ action on rational maps, i.e. the correspondence is equivariant. We then show that this is the same map as given by Theorem A and we find that real triviality (3.10) is implied by the other monad conditions in the $\text{Harm}(\mathbb{S}^2, U(2))$ case. This implies that $U(2)$ monads always have $\gamma = 0$. Of course, this is not the way we would like to prove it.

10.1 Rational maps. Since harmonic maps into $U(2)$ are 1-unitons, they have the form $S = Q(\pi - \pi^\perp)$, for some $Q \in U(2)$, and π , projection onto a holomorphic subbundle of $\underline{\mathbb{C}}^N$.

We can see the decomposition $S = Q(\pi - \pi^\perp)$ as a composition of three maps. In the middle is the Cartan embedding

$$\mathbb{P}^1 = \text{Gr}_1(\mathbb{C}^2) \xrightarrow{I} U(2) : \pi \in \text{Gr}_1(\mathbb{C}^2) \mapsto \pi - \pi^\perp \in U(2).$$

In terms of $z \in \mathbb{P}^1 = \text{Gr}_1(\mathbb{C}^2)$,

$$\begin{aligned} \pi_z &= \frac{1}{1 + z\bar{z}} \begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix}^*, \\ \pi_z^\perp &= \frac{1}{1 + z\bar{z}} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix}^*, \end{aligned}$$

so

$$(10.2) \quad I(z) = \pi_z - \pi_z^\perp = \frac{1}{1 + z\bar{z}} \begin{pmatrix} 1 - z\bar{z} & 2\bar{z} \\ 2z & z\bar{z} - 1 \end{pmatrix}.$$

All holomorphic subbundles, $\pi \subset \underline{\mathbb{C}}^2$, are π_f for some rational function $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Finally, we can left-translate harmonic maps $L_Q : S \mapsto QS$. In other words, a general one uniton is a composition

$$\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1 \xrightarrow{I} U(2) \xrightarrow{L_Q} U(2)$$

where $f \in \text{Rat } \mathbb{P}^1$ and $Q \in U(2)$. To what extent is it unique?

The group $U(2)$ acts on itself by left translation, and in turn on the totally-geodesically-embedded spheres. The translates of $I(\mathbb{P}^1)$ are $U(2)/\text{Stab}_{I(\mathbb{P}^1)} U(2)$. Since $I(\mathbb{P}^1) = \{S \in U(2) : S^2 = \mathbb{I}\}$, and $\mathbb{I} = (QI(0))^2 = (QI(1))^2 = (QI(i))^2$ iff $Q = \{\pm \mathbb{I}\}$, $\text{Stab}_{I(\mathbb{P}^1)} U(2) = \{\pm \mathbb{I}\}$. So left translation on embedded spheres is not faithful. Since $(-\mathbb{I})I(z) = I(-1/\bar{z})$ is orientation reversing, it is faithful on oriented spheres, and hence on $\text{Harm}(\mathbb{S}^2, U(2))$, i.e. the decomposition $S = Q(\pi - \pi^\perp) : (U(2) \times \text{Rat}(\mathbb{P}^1)) \rightarrow \text{Harm}(\mathbb{S}^2, U(2))$ is unique.

Rational maps can be written as $p(z)/q(z)$, with $(p, q) = 1$. Their topological degree is given by $\max\{\deg p, \deg q\}$ (always positive, because holomorphic maps preserve orientation). They contain the based maps

$$\text{Rat}^* = \{f \in \text{Rat} : f(\infty) = 0\} = \left\{ \frac{p(z)}{q(z)} : (p, q) = 1, \deg p < \deg q \right\}.$$

Various groups act on Rat via the action of $\text{PGL}(2)$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \frac{p}{q} \mapsto \frac{Ap + Bq}{Cp + Dq}.$$

This map preserves degree because $\text{GL}(2)$ is connected and degree components are disjoint. Rat is a Rat^* bundle over \mathbb{P}^1 , given by

$$\text{rat} : \text{Rat} \rightarrow \mathbb{P}^1 : f \mapsto f(\infty).$$

$\text{PGL}(2)$ acts on \mathbb{P}^1 by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : [x, y] \mapsto [Ax + By, Cx + Dy],$$

making $\text{Rat} \rightarrow \mathbb{P}^1$ an equivariant bundle.

$$\{P \in \text{PGL}(2) : P(\text{Rat}^*) = \text{Rat}^*\} = \left\{ \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix} \right\}.$$

Conjugation acts on $\text{Harm}^*(\mathbb{S}^2, \text{U}(2))$ and the isomorphism $\text{Harm}^*(\mathbb{S}^2, \text{U}(2)) \cong \text{Rat } \mathbb{P}^1$ is equivariant.

Claim 10.3. *If $p/q \in \text{Rat } \mathbb{P}^1$ and $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{U}(2)$ then*

$$(10.4) \quad UI(p/q)U^* = I \left(\frac{Cq + Dp}{Aq + Bp} \right).$$

Hint. Write

$$I(p/q) = \frac{1}{p\bar{p} + q\bar{q}} \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}^*. \quad \square$$

10.5 Monads. We will now relate monad description to the rational maps description. The second Chern class gives a stratification

$$\text{Harm}^*(\mathbb{S}^2, \text{U}(2)) = \bigcup_{k/2 \in \mathbb{N}} \text{Harm}_{k/2}^*(\mathbb{S}^2, \text{U}(2)).$$

(The quantity $k/2$ is also the jumping type of $\mathcal{V}|_{P_0}$). Degree gives a stratification

$$\text{Rat} = \bigcup_j \text{Rat}_j.$$

The map $\text{Harm}(\mathbb{S}^2, \text{U}(2))/\text{U}(2) \rightarrow \text{Rat}$ given by the formula of Theorem A preserves this stratification.

Recall the Jordan block normalisation of a $\gamma = 0$ monad M , given by the data α'_1, a', b' . From the monad equation, we saw that $b'a' = 0$, but both b' and a' must be nonzero if J and K are to be injective and surjective respectively. It follows that the column space of a' and row space of b' are one dimensional, and that they are perpendicular to one another with respect to the Euclidean metric on \mathbb{C}^2 . From the discussion of the normalisation, we see that the Jordan blocks of α'_1 have distinct eigenvalues, and the monad will be given by

$$(10.6) \quad \begin{aligned} \alpha'_1 &= \begin{pmatrix} \text{Jordan}(j_1, e_1) & & \\ & \ddots & \\ & & \text{Jordan}(j_L, e_L) \end{pmatrix}, \\ a' &= \begin{pmatrix} 1 \\ c_1 \end{pmatrix} (1, 0, \dots, 0, 1, 0, \dots), \\ b' &= \tilde{b} \begin{pmatrix} -c_1 & 1 \end{pmatrix} \end{aligned}$$

generically, where $\tilde{b}_{j_1}, \tilde{b}_{j_1+j_2}, \dots$ are not zero, and the first, $j_1 + 1\text{st}, \dots$ columns of a are not zero.

When $c_1 = 0$,

$$(10.7) \quad \frac{1}{2} a' (\alpha'_1 - 2z)^{-1} b' = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$$

where

$$(10.8) \quad f(z) = - \sum_{i=1}^L \sum_{j=1}^{j_i} \frac{\tilde{b}_{j_1+\dots+j_i+j} (z - e_i)^{j_i-j}}{(z - e_i)^{j_i}} = \frac{p}{q}$$

is a based rational map of degree $k/2$. We see that

$$S = \frac{1}{1 + f\bar{f}} \begin{pmatrix} 1 - f\bar{f} & 2f \\ -2\bar{f} & 1 - f\bar{f} \end{pmatrix}.$$

This is equivalent to (10.2) under a change of frame (basing condition).

Remark 10.9. We see from this calculation that *all* $U(2)$ unitons have $\gamma = 0$, as expected.

Remark 10.10. Putting this into the determinant form of real triviality (3.10), we can

$$\begin{aligned} (3.10) &= \prod_{i=1}^L |(e_i + z)|^{2j_i} \left(1 + \left| \sum_{i=1}^L \sum_{j=1}^{j_i} (e_i + z)^{-j} \tilde{b}_{j_1+\dots+j_i+j} \right|^i \right) \\ &= |q|^2 + |p|^2 \end{aligned}$$

where $f = p/q$ as above, which is not zero since $(p, q) = 1$ by construction.

Remark 10.11. The interpretation of the determinant condition $(3.10) = |q|^2 + |p|^2$ tells us what happens (in the $\gamma = 0$ case) when it fails. The points where it fails are the common zeros of p and q . Nondegenerate monads do not give rise to such polynomials. In the degenerate case, i.e. on part of the boundary, the poles and zeros of f coalesce and the holomorphic map changes degree as the harmonic map undergoes bubbling off.

Remark 10.12. Having worked out the simple $U(2)$ case in detail, we can already give an interesting example ([BuGu]). One-unitons are known to be holomorphic or antiholomorphic. The simplest such solution is $I(z)$ in our notation, which is the map into \mathbb{P}^1 associated to the tautological subbundle of $\mathbb{C}_{\mathbb{P}^1}^2$. The orthogonal complement of this bundle is represented by $-I(z)$. The subbundle of $\mathbb{C}_{\mathbb{P}^1}^4$, which is the sum of the tautological bundle in the first \mathbb{C}^2 and its complement in the second \mathbb{C}^2 , is not holomorphic in $\text{Gr}_2(\mathbb{C}^4)$, so one might expect it to be a two-uniton. As a map into $U(4)$, however, it is equivalent, under left multiplication by a scalar, to a holomorphic map into $\text{Gr}_2(\mathbb{C}^4)$. In fact, the uniton bundle decomposes into two copies of the simplest nontrivial bundle, and hence must be a 1-uniton. This also tells us that the map has energy 2.

11. GRASSMANNIAN SOLUTIONS

The involution $S \mapsto S^{-1}$ is an involution of $U(N)$, whose fixed set has $N-1$ components corresponding to the various Grassmann manifolds of subspaces of \mathbb{C}^N . This fixed set is the image of the Cartan embedding which takes a subspace to the difference of projection onto the subspace and projection onto its orthogonal complement. Because this is a totally geodesic embedding, harmonic maps into Grassmannians are in bijection with unitons whose image lies in the fixed set of $S \mapsto S^{-1}$. To be able to say something about these maps, we will restrict ourselves to based maps. The usual basing condition, $S(\infty) = \mathbb{I}$, does not make sense for such maps. We must choose different basepoints in each component,

$$S(\infty) = \begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix}$$

where the signature of the basepoint distinguishes the image Grassmannian. We have described an inclusion

$$\text{Harm}^*(\mathbb{S}^2, \text{Gr}_j(\mathbb{C}^N)) \hookrightarrow \text{Harm}^*(\mathbb{S}^2, U(N)).$$

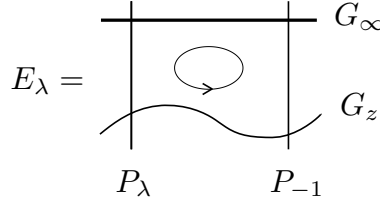
We would like to identify intrinsically which uniton bundles correspond to based maps into Grassmannians.

Let $S \in \text{Harm}^*(\mathbb{S}^2, \text{Gr}_j(\mathbb{C}^N))$ and let E_λ be an extended solution for S and assume that $E_\lambda(\infty) = \begin{pmatrix} \mathbb{I} \\ -\lambda \mathbb{I} \end{pmatrix}$. Uhlenbeck shows that $\tilde{E}_\lambda = E_{-\lambda} E_1^{-1}$ is an extended solution for S^{-1} . If $S = S^{-1}$, then these extended solutions are equivalent (after left multiplication by a function of λ), and hence determine the same uniton bundle, up to framing.

Now define an involution by

$$(11.1) \quad \mu^* \lambda = -\lambda, \quad \mu^* \eta = \eta \quad (\text{equivalently } \mu^* z = z),$$

and fix generators of $\pi_1(G_\infty \cup P_{-1} \cup G_z \cup P_\lambda)$ above which we calculate the monodromies which determine E_λ :



Then the formula for \tilde{E}_λ has the interpretation

$$\begin{aligned} E_{-\lambda} E_1^{-1} &= \begin{array}{c} \text{Diagram: Rectangle with } G_\infty \text{ at top, } G_z \text{ at bottom, } P_{-\lambda} \text{ on left, } P_{-1} \text{ on right. Circular arrow with right-pointing arrowhead.} \end{array} \circ \begin{array}{c} \text{Diagram: Rectangle with } G_\infty \text{ at top, } G_z \text{ at bottom, } P_1 \text{ on left, } P_{-1} \text{ on right. Circular arrow with left-pointing arrowhead.} \end{array} \\ &= \begin{array}{c} \text{Diagram: Rectangle with } G_\infty \text{ at top, } G_z \text{ at bottom, } P_\lambda \text{ on left, } P_{-1} \text{ on right, and an additional vertical line } P_1 \text{ on the far right. Two circular arrows, both with right-pointing arrowheads, are shown.} \end{array} \\ &= \begin{array}{c} \text{Diagram: Rectangle with } G_\infty \text{ at top, } G_z \text{ at bottom, } P_{-\lambda} \text{ on left, } P_1 \text{ on right. Circular arrow with right-pointing arrowhead.} \end{array} = \mu^* \left(\begin{array}{c} \text{Diagram: Rectangle with } G_\infty \text{ at top, } G_z \text{ at bottom, } P_\lambda \text{ on left, } P_{-1} \text{ on right. Circular arrow with right-pointing arrowhead.} \end{array} \right) = \mu^* E_\lambda. \end{aligned}$$

Since \tilde{E}_λ determines the uniton bundle and vice versa, the uniton bundle for S^{-1} is the μ -pullback of the bundle for S , up to the choice of framing. To see the effect on the framing, note that when $E_\lambda(\infty) = \begin{pmatrix} \mathbb{I} \\ -\lambda \mathbb{I} \end{pmatrix}$ (the difference between the chosen framing of $\mathcal{V}|_{G_\infty}$ and the canonical one) μ carries a frame at $P_{-1} \cap G_\infty$ to $\begin{pmatrix} \mathbb{I} \\ -\mathbb{I} \end{pmatrix}$ times itself (after ‘transporting’ it back using evaluation of the canonical frame).

So uniton bundles (\mathcal{V}, Φ) correspond to Grassmannian solutions iff μ lifts to $\tilde{\mu} : \mathcal{V} \rightarrow \mathcal{V}$ and the signature of $\mu^* \phi \phi^{-1} : \mathbb{C}^N \rightarrow \mathbb{C}^N$ determines the component (rank of the image Grassmannian).

11.2 Monad characterisation. The bundle lift $\tilde{\mu}$ is a bundle map $\mathcal{V} \rightarrow \mu^*\mathcal{V}$, but we want more than this, namely a map of monads. On \mathbb{P}^2 monads are mapped to monads by linear maps. The induced action of μ is linear if we take the birational equivalence from §8, i.e. the one in which we blow up and down on P_∞ . The induced action is

$$\begin{aligned} X &\mapsto -X \\ \mu : \quad Y &\mapsto Y \\ W &\mapsto W. \end{aligned}$$

As in the case of time invariance, this means we must work with half the bundle. In what has gone before it has been convenient to frame our bundles above P_{-1} . To see that μ -invariance is a local property one should think of bundles framed over G_∞ . The involution $\tilde{\mu}$ may as well be thought of as an involution of the space of frames of a given bundle.

If (J, K) is a normalised monad representing $\mathcal{V}_{\text{north}}$

$$(11.3) \quad \begin{aligned} J_W &= \begin{pmatrix} \mathbb{I} \\ 0 \\ 0 \end{pmatrix} & J_X - J_Y &= \begin{pmatrix} 0 \\ \mathbb{I} \\ 0 \end{pmatrix} & J_X + J_Y &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix} \\ K_W &= (0 \quad \mathbb{I} \quad 0) & K_X - K_Y &= (-\mathbb{I} \quad 0 \quad 0) & K_X + K_Y &= (-\alpha_2 \quad \alpha_1 \quad b) \end{aligned}$$

(primes omitted) then the pulled-back monad

$$(11.4) \quad \begin{aligned} \mu^* J_W &= \begin{pmatrix} \mathbb{I} \\ 0 \\ 0 \end{pmatrix} & \mu^* J_{X-Y} &= \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ -a \end{pmatrix} & \mu^* J_{X+Y} &= \begin{pmatrix} 0 \\ -\mathbb{I} \\ 0 \end{pmatrix} \\ \mu^* K_W &= (0 \quad \mathbb{I} \quad 0) & \mu^* K_{X-Y} &= (\alpha_2 \quad -\alpha_1 \quad -b) & \mu^* K_{X+Y} &= (\mathbb{I} \quad 0 \quad 0) \end{aligned}$$

represents $\mu^*\mathcal{V}$. If $G \in \text{Gl}(k/2)$, $F \in \text{Gl}(N)$, then

$$(11.5) \quad G \quad \begin{pmatrix} G & -G\alpha_2^{-1}\alpha_1 & -G\alpha_2^{-1}b \\ & -G\alpha_2^{-1} & \\ & -Fa\alpha_2^{-1} & F \end{pmatrix} \quad G\alpha_2^{-1}$$

(acting linearly on the three homogeneous bundles of the monad) renormalises $\mu^*(J, K)$. From this we deduce that $\mathcal{V} \cong \mu^*\mathcal{V}$ iff there exist G and F such that

$$(11.6) \quad \begin{aligned} G\alpha_2^{-1}\alpha_1 G^{-1} &= \alpha_1 & Fa\alpha_2^{-1}G^{-1} &= a \\ -G\alpha_2 G^{-1} &= \alpha_2 & -G\alpha_2^{-2}bF^{-1} &= b, \end{aligned}$$

(primes omitted) and F represents the action on the framing; in our case we require $F^2 = \mathbb{I}$.

Remark 11.7. In the case of 1-unitons, α_2 is diagonal and the monad equation reduces to $ba = 0$. Letting F act on $\ker b$ by $-\mathbb{I}$ and on the complement by \mathbb{I} , and taking $G = \mathbb{I}$, we see that all (based) 1-unitons come from (based) maps into Grassmannians.

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MATHEMATICS RESEARCH CENTRE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL

E-mail address: `anand@maths.warwick.ac.uk`